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ON NORMALS TO CONICS, A NEW TREATMENT OF THE SUBJECT.

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LET $a\,a'\,b\,b'\,c\,c'$ be the vertices of a quadrilateral whose diagonals aa', bb', cc' form the triangle $\alpha\beta\gamma$. Any line R intersects the diagonals in three points whose harmonic conjugates relative to the couples aa', bb', cc', respectively, lie on another line R', which may be said to *correspond* to R *). The four sides of the quadrilateral are the only lines which coincide with their corresponding ones. When R passes through a vertex of the quadrilateral, R' passes through the same vertex, and the two lines are harmonic conjugates relative to the sides which intersect at that vertex. When R coincides with a diagonal, R' is an indeterminate line passing through the intersection of the other two diagonals.

When R turns around a fixed point p, R' envelopes a conic P inscribed to the triangle $\alpha\beta\gamma$, and obviously identical with the envelope of the polars of p relative to the several conics inscribed in the quadrilateral. Hence it follows that the tangents from p to P form a pair of corresponding lines, and that they are the tangents at p to the two inscribed conics which pass through the latter point.

*) This property is easily demonstrated; it is in fact a particular case of a more general theorem due to HESSE (CRELLE'S *Journal*, vol. XX.) in virtue of which any three pairs of harmonic conjugates relative to aa', bb', cc' lie on a conic. A geometrical demonstration of this more general theorem is also given by CHASLES at p. 96, of the first part of his excellent *Traité des Sections Coniques*. T. A. H.

In a similar manner, the harmonic conjugates relative to aa', bb', cc' of the six intersections of the diagonals with any conic C lie on a second conic C'. If the former be conjugate to the triangle $\alpha\beta\gamma$, so also will the latter, and the two conics C, C', in this case, will be the reciprocal polars relative to the fourteen-points conic (p. 13) [Queste Opere, n. **64**]. L. C.

Conversely, when R envelopes a conic P inscribed to the triangle $\alpha\beta\gamma$, its corresponding line R′ always passes through a fixed point p *corresponding* to that conic.

When p is on a diagonal, the conic P resolves itself into a pair of points of which one coincides with the intersection of the other two diagonals, and the other with the harmonic conjugate of p relative to the vertices situated on the first diagonal.

In this manner we have a method of transformation in which to a line corresponds a line, and to a point corresponds a conic inscribed in a fixed triangle $\alpha\beta\gamma$. It can, moreover, be shown, that to a curve of m^{th} class which touches the sides of this triangle in λ, μ, ν points, respectively, corresponds a curve of the class $2m-(\lambda+\mu+\nu)$, having these sides for multiple tangents of the orders

$$m-(\mu+\nu),\ m-(\nu+\lambda),\ m-(\lambda+\mu).\ ^{*)}$$

If the points c, c' coincide with the imaginary circular points at infinity, the inscribed conics will form a system of confocal conics; a, a' and b, b' being their common foci (real and imaginary), and γ their common centre.

Corresponding lines R, R′ are now perpendicular to each other, and divide harmonically the focal segments aa', bb'. Two such lines, therefore, are necessarily tangent and normal to each of the two confocal conics passing through their intersection. In other words, any line R whatever being regarded as a tangent (or as a normal) at one of its points to a determinate conic of the confocal system, the corresponding line R′ will be the normal (or the tangent) to that conic at that point.

To a point p corresponds a parabola P touching the axes aa', bb' and having the line $p\gamma$ for directrix.

To the normals which can be drawn from p to any conic C of the confocal system, correspond the tangents common to C and to the parabola P; so that the problem to draw the normals from a point p to a given conic C, is transformed to this: to find the common tangents to a conic C and a parabola P, which touches the axes of C as well as the bisectors of the angle subtended at p by C. The four common tangents being constructed, the required normals will be the lines joining p to their points of contact with C. The anharmonic ratio of the four normals, it may be added, is equal to that of the four tangents.

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*) A similar method of transformation is given by STEINER in his *Geometrische Gestalten*, p. 277, and a precisely correlative method has been investigated by Prof. H. A. NEWTON (*Math. Monthly,* Vol. III. p. 235), and by Prof. BELTRAMI (*Mem. dell'Accad. delle Scienze di Bologna,* Ser. II. Vol. II.). Formulæ analogous to the above are also given in my paper « On the Quadric Inversion of Plane Curves » (Proc. of R. S. March, 1865) the effects of such inversion being the same as those of the transformations of Professors NEWTON and BELTRAMI. T.A.H.

The feet of the four normals are the intersections of C, and the conic H which is the reciprocal polar of P relative to C. Now P being inscribed to a triangle $\alpha\beta\gamma$ which is conjugate to C, H will be circumscribed to this triangle; that is to say, it will be an equilateral hyperbola passing through the centre of C, and having its asymptotes parallel to the axes of C. Moreover H is intersected by the polar of p, relative to C in two points, conjugate with respect to C, whose connector subtends a right angle at p.

Conversely, every equilateral hyperbola H circumscribed to $\alpha\beta\gamma$ will intersect C in four points, the normals (to C) at which will converge to a point p; in fact, to that point which corresponds to the parabola P of which H is the polar reciprocal, relative to C.

Since to the several tangents of any conic C of the confocal system correspond the normals at the points of contact; the curve corresponding to C itself will be its envolute E; which, by the above, must be a curve of the *fourth* class, having for double tangents the axes aa', bb' of C and the line cc' at infinity; moreover, E will touch C at the four imaginary points where the latter touches the sides of the quadrilateral whose six vertices are the four foci a, a', b, b', and the two circular points c, c'. To the several points of E correspond parabolas P which touch C; hence, since there are four parabolas P which have double contact with C, E has four double points. Further, the points will be stationary ones on E, which correspond to parabolas P having three-pointic contact with C. But to possess this property such a parabola must necessarily resolve itself into a vertex of the triangle $\alpha\beta\gamma$, and an intersection of the opposite side with the conic C. Hence E has six cusps e, e'; f, f'; g, g'; situated, two and two, on the sides of the triangle $\alpha\beta\gamma$; they are in fact, the harmonic conjugates, relative to the vertices aa', bb', cc', of the intersections of C with the sides of $\alpha\beta\gamma$. Hence it follows (see note) first, that the six cusps lie on the conic C', which constitutes the polar reciprocal of C relative to the fourteen-points conic; secondly, that E is a curve of the sixth order; and thirdly, that this curve is touched by its double tangents aa', bb', cc' precisely at its cusps.

This will suffice to show with what facility questions concerning normals to conics may be treated by the above method, and how by its means the numerous theorems due to Poncelet, Chasles, Joachimsthal, and others; as well as the more recent theorems of Steiner (Crelle's *Journal*, Vol. XLIX.) and Clebsch (*Ibid.* Vol. LXII.) may be rendered geometrically evident.

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