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ON A GEOMETRICAL TRANSFORMATION OF THE FOURTH ORDER IN SPACE OF THREE DIMENSIONS, THE INVERSE TRANSFORMATION BEING OF THE SIXTH ORDER.

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The subject of this short Memoir is that of Geometrical Rational Transformation in Space of Three Dimensions. When the points of a space S have a (1, 1) correspondence with those of another space s, in such a manner that the planes and the (right) lines of s correspond to surfaces F of m^{th} order, and to curves C of the n^{th} order in the former space S, I say that the transformation of s into S is of the m^{th} degree, and that the inverse transformation (of S into s) is of the n^{th} degree. In fact the planes and the (right) lines of S correspond to surfaces Φ of the n^{th} order, and to curves k of the m^{th} order, in the space s. The surfaces F (or Φ) are homaloid (unicursal after Cayley and Salmon), and form what I name a *homaloidic* system; that is to say, a single surface will pass through three *arbitrary* points, and three surfaces will have a single common point (if not an infinite number), wich is not common to the whole system. Two surfaces have, in common, a unicursal curve C (or k).

Generally there are (*fundamental*) points common to all the surfaces F (or Φ); if their number is an infinite one, they form (*fundamental*) curves, which can be single or multiple for the surfaces of the system. The points on fundamental curves of each system are *exceptional* ones, with respect to the transformation: they do not correspond to *single* points of the other space, but rather to curves, whose loci are surfaces forming the *Jacobiana* of the second system. Of the two homaloidic systems, if one is known, the other is determined too. And if the correspondence is given betwen the points of a single homaloid surface and those of a plane, my method finds out all the homaloidic systems to which the given surface can belong.

I will explain the method by an example, which may be considered as possessing some interest in itself. Let us start from a Nöther's surface F of the 4th order, whose only singularity is a double point O with a single tangent plane Ω, which meets the surface in four (right) lines passing through O. This surface may be representend on a plane f in such a manner that the images of its plane sections may be sextic curves a^2b, having seven double points $a_1, a_2, \dots a_7$, and four single points b_1, b_2, b_3, b_4 common: all these eleven points being ranged in a single cubic curve ab, which is the image of the singular point O. *)

Any other surface F', having the same singularity as F (the same double point O, and the same singular tangent plane Ω), will intersect F in a curve of the 16th order, whose image on the plane f is a curve a^4 of the 12th order, passing four times through each of the seven points a.

In several ways the curve of the 12th order can break up into others of lower degree: I assume a braking up into a right line and a curve a^4 of the 11th order. Supposing this latter as fixed or given, and the former as variable on the plane f, we shall have the images of ∞^2 unicursal curves C of the 6th order, in which F is cut by ∞^3 surfaces F' passing (like F) through a common curve Γ of the 10th order, whose deficiency is 3. By consideration of images in f, we see that O is a treble point for the curves C, and a fivefold one on Γ; and that every C meets Γ in eleven points.

Thus we have got a homaloidic system of surfaces F, F', ... to which the given Nöther's surface belongs. Consequently there exists a (1, 1) correspondence between the points, the planes f and the lines c of a space s, and the points, the quartic surfaces F, and the sextic curves C of the space S, from which we started. Then the planes Φ and the lines K of S will correspond to surfaces φ of the 6th order and to curves k of the 4th order; for, the intersections of a φ with a line, or of a k with a plane, in s, must correspond to the intersections of a plane Φ with a sextic curve C, or of a line K with a quartic surface F in S.

Besides the variable intersection k of the 4th order, the surfaces φ of the homaloidic system in s must have in common some fixed curves equivalent to one of the order $(6^2 - 4 =)32$. To find them out, let us remember that a point of a r-ple curve common to the φ's corresponds to a unicursal curve of the r^{th} order lying on the Jacobiana of the F's, and that the intersections of the r-ple curve with a plane f correspond to an equal number of curves of the r^{th} order common to an F and to the Jacobiana.

The curves of F corresponding to single points in f are seven conics and four lines whose images are the seven points a and the four points b; therefore the surfaces φ of the homaloidic system in s will have in common a double curve δ of the 7th order, and a simple curve ω of the 4th order. The Jacobiana of the F's is then composed of two parts:

*) See a Paper of mine in the *Collectanea mathematica in memoriam D. Chelini*. [Queste Opere, n. **108**].

1.° the surface Δ of the 11th order, which is the locus of the conics touching in O the plane Ω and meeting the curve Γ four times; 2.° the plane Ω as being the locus of lines through O. The curve Γ is a treble one on Δ.

An arbitrary plane Φ in S corresponds to a sextic surface φ in s; but, if Φ is a plane Ψ passing through O, which is a treble point on any sextic curve C, the correspondent φ will break into two cubic surfaces; of which the one, ψ_0, is fixed and answers to the fundamental point O; whereas the other, ψ, is variable in a linear two-fold system (*réseau*) including ψ_0. The cubic ψ coincides with ψ_0 when the plane Ψ falls on Ω. As on every φ, δ is a double curve of the 7th order and ω a simple one of the 4th order, we see at once that all the cubic surfaces ψ pass through δ, but only ψ_0 passes through ω too.

Reciprocally all cubic surfaces (ψ) through the fundamental curve δ of the 7th order correspond to planes (Ψ) through O: because the partial Jacobiana Δ must then break off from the surface of the order (3 . 4 =) 12 corresponding (in S) to any cubic surface (in s).

Let us consider the representation of the points of any ψ on those of its correspondent plane Ψ. This plane meets Γ in five points, besides O; therefore it will cut the F's in curves of the 4th order, of which $4^2 - 5 - 3 = 8$ common points are coinciding with O. Consequently, the images of plane sections of ψ on Ψ are quartic curves (with two consecutive double points on O and Ω) having eight common intersections gathered up in O and five other common (simple) points. This threefold system of quartics contains a twofold one of such curves having a treble point at O (with two coinciding tangent lines on Ω); whence it follows that the cubic surfaces ψ possess a common double point o.

Two cubic surfaces ψ meet, further, in a conic corresponding to a line through O. Therefore all the analogous conics are passing through o, and this point is a treble one on the curve δ, whose deficiency is 3. This can be proved by projection of one ψ from the double point o on a plane π; a representation which can also result from the foregoing one of ψ on Ψ, combined with quadric transformation. The images of plane sections are then cubic curves meeting in six fixed points p the conic which represents the double point o; and the image of the intersection of our surface with any other cubic surface [113] will be a quintic p. A conic passing through o is represented by a line drawn through one of the points p; then, if the conic is common to the two cubic surfaces, the image of their further intersection δ will be a quartic passing through the five remaining points p; whence we recognize at once the deficiency of δ and the number of its branches through o. Besides, we see that the conic meets δ in four points in addition to o.

Any plane Φ (in S) intersects the F's in a threefold system of quartic curves passing through ten common points G, which are the tracks of Γ; these curves are then the images of plane sections of a surface φ, when this is put in correspondence with the plane Φ.

Looking up to the intersection of this plane with the Jacobiana of the F's, we find that the fundamental curves δ, ω have, as their images on Φ, a curve $\nabla \equiv G^3$ of the 11th order and a right line, whence it follows that δ and ω meet in eleven points. Any one of the conics, whose locus is the partial Jacobiana Δ, pierces the plane Φ in two *conjugate* points of ∇ answering to one point of the double fundamental curve δ. Every quartic G meets with ∇ only in pairs of conjugate points.

From this representation on Φ, we infer that every sextic surface φ contains ten right lines, which are lines through three points of the double curve δ. The locus of such lines is a partial Jacobiana for the φ's corresponding to Γ: its degree is 11; for, in the space S, every curve C meets Γ in 11 points. The fundamental point O is a treble one for every C, and absorbs nine of the twenty-four linear conditions determining a unicursal twisted sextic curve; therefore, the Jacobiana of the φ's is completed by a surface of the third order three times counted: the cubic ψ_0. We see at once that δ is a 4-ple curve on the former partial Jacobiana.

Any two of the φ's have as intersection (besides δ and ω) a unicursal quartic curve k, corresponding to a right line in S; every k meets δ in eleven points, and ω in one point, because in S these numbers eleven and one are the orders of the surfaces making up the Jacobiana of the homaloidic system.

Applying to the projection of ψ_0 from o on a plane, it is easy to see that o is a treble point on every surface φ.

We saw above that lines K (in S) correspond to quartic curves k meeting ω in one and δ in eleven points; and that lines c (in s) correspond to sextic curves C cutting Γ eleven times and having O as treble point with tangent lines on the plane Ω. But curves k and C will degenerate into curves of lower order when the line K meets Γ or passes through O, or when the line c cuts δ or ω; for, to any point of Γ, ω, δ, corresponds a line or a conic, &c. Thus, for instance, lines cutting Γ three times correspond to chords of δ supported by ω; lines projecting Γ from O correspond to lines projecting δ from o; lines through three points of ω correspond to cubic (twisted) curves meeting Γ in eleven points; &c. The point o has, as its image in the space S, the three chords of Γ which spring from O.

Our transformation, by which we purposed to give an example of the general method, is now established and completely explained in its chief circumstances.